Exercise 8

A spring has a mass of 1 kg and its damping constant is $c = 10$. The spring starts from its equilibrium position with a velocity of 1 m/s. Graph the position function for the following values of the spring constant k: 10, 20, 25, 30, 40. What type of damping occurs in each case?

Solution

The equation of motion for a mass attached to a spring and a dashpot is

$$-c\frac{dx}{dt} - kx = m\frac{d^2x}{dt^2}.$$

Bring all terms to the left side.

$$m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = 0 \tag{1}$$

This is a linear homogeneous ODE, so its solutions are of the form $x = e^{rt}$.

$$x = e^{rt} \quad \rightarrow \quad \frac{dx}{dt} = re^{rt} \quad \rightarrow \quad \frac{d^2x}{dt^2} = r^2e^{rt}$$

Plug these formulas into equation (1).

$$m(r^2e^{rt}) + c(re^{rt}) + k(e^{rt}) = 0$$

Divide both sides by e^{rt}.

$$mr^2 + cr + k = 0 \tag{2}$$

Overdamping occurs when $c^2 - 4mk > 0$, that is, when $k = 10$ and $c = 20$. In this case, the solution to equation (2) is

$$r = \frac{-c \pm \sqrt{c^2 - 4mk}}{2m}$$

$$r = \left\{\frac{-c - \sqrt{c^2 - 4mk}}{2m}, \frac{-c + \sqrt{c^2 - 4mk}}{2m}\right\}.$$

Two solutions to the ODE are

$$\exp\left(\frac{-c - \sqrt{c^2 - 4mk}}{2m}t\right) \quad \text{and} \quad \exp\left(\frac{-c + \sqrt{c^2 - 4mk}}{2m}t\right).$$

By the principle of superposition, then, the general solution to equation (1) is

$$x(t) = C_1 \exp\left(\frac{-c - \sqrt{c^2 - 4mk}}{2m}t\right) + C_2 \exp\left(\frac{-c + \sqrt{c^2 - 4mk}}{2m}t\right),$$

where C_1 and C_2 are arbitrary constants. Differentiate it with respect to t to get the velocity.

$$\frac{dx}{dt} = C_1\frac{-c - \sqrt{c^2 - 4mk}}{2m} \exp\left(\frac{-c - \sqrt{c^2 - 4mk}}{2m}t\right) + C_2\frac{-c + \sqrt{c^2 - 4mk}}{2m} \exp\left(\frac{-c + \sqrt{c^2 - 4mk}}{2m}t\right)$$

Apply the initial conditions, $x(0) = 0$ and $x'(0) = 1$, to determine C_1 and C_2.

$$x(0) = C_1 + C_2 = 0$$

$$\frac{dx}{dt}(0) = C_1 \frac{-c - \sqrt{c^2 - 4mk}}{2m} + C_2 \frac{-c + \sqrt{c^2 - 4mk}}{2m} = 1$$

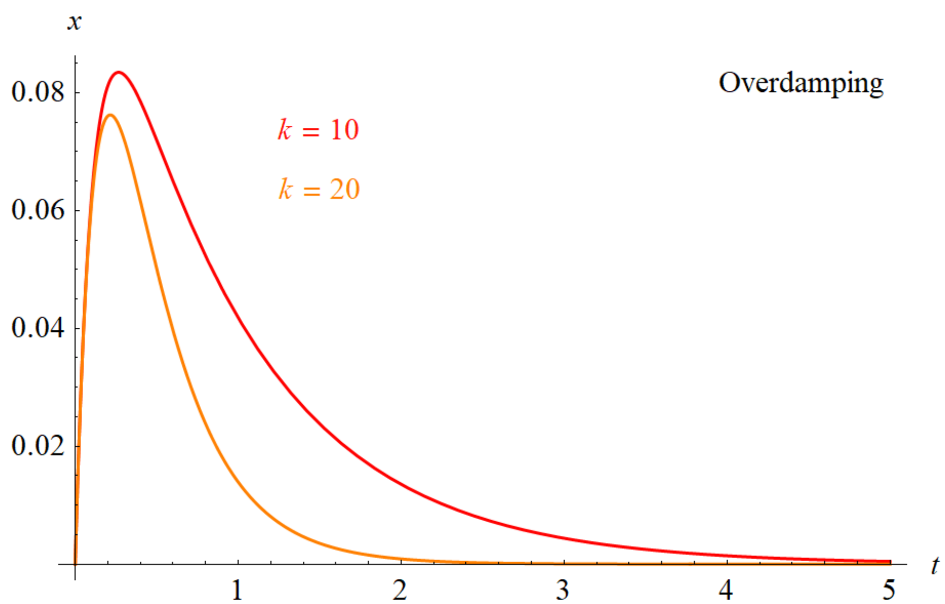
Solving this system of equations yields

$$C_1 = -\frac{m}{\sqrt{c^2 - 4mk}} \quad \text{and} \quad C_2 = \frac{m}{\sqrt{c^2 - 4mk}},$$

meaning the displacement from equilibrium in overdamping is

$$\boxed{x(t) = -\frac{m}{\sqrt{c^2 - 4mk}} \exp\left(\frac{-c - \sqrt{c^2 - 4mk}}{2m} t\right) + \frac{m}{\sqrt{c^2 - 4mk}} \exp\left(\frac{-c + \sqrt{c^2 - 4mk}}{2m} t\right).}$$

Plug in $m = 1$ kg and $c = 10$ N/m and graph $x(t)$ versus t for $k = 10$ N · s/m and $k = 20$ N · s/m.



Critical damping occurs when $c^2 - 4mk = 0$, that is, when $k = 25$. In this case, the solution to equation (2) is

$$r = -\frac{c}{2m}$$

$$r = \left\{-\frac{c}{2m}\right\}.$$

Two solutions to the ODE are

$$\exp\left(-\frac{c}{2m}t\right) \quad \text{and} \quad t \exp\left(-\frac{c}{2m}t\right).$$

By the principle of superposition, then, the general solution to equation (1) is

$$\begin{aligned} x(t) &= C_3 \exp\left(-\frac{c}{2m}t\right) + C_4 t \exp\left(-\frac{c}{2m}t\right) \\ &= \exp\left(-\frac{c}{2m}t\right)(C_3 + C_4 t), \end{aligned}$$

where C_3 and C_4 are arbitrary constants. Differentiate it with respect to t.

$$\frac{dx}{dt} = -\frac{c}{2m}\exp\left(-\frac{c}{2m}t\right)(C_3 + C_4 t) + \exp\left(-\frac{c}{2m}t\right)(C_4)$$

Apply the initial conditions, $x(0) = 0$ and $x'(0) = 1$, to determine C_1 and C_2.

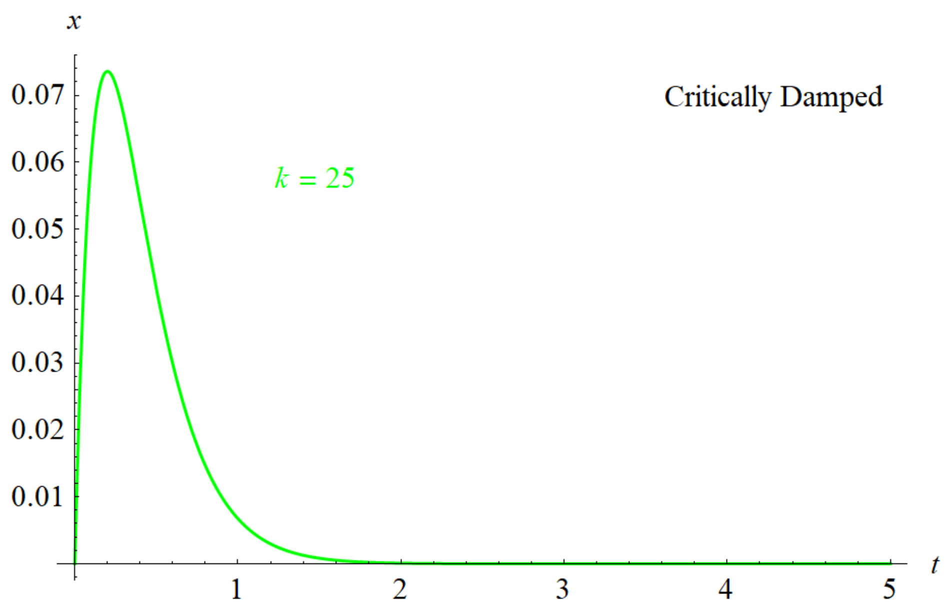
$$x(0) = C_3 = 0$$

$$\frac{dx}{dt}(0) = -\frac{c}{2m}C_3 + C_4 = 1$$

Solving this system of equations yields $C_3 = 0$ and $C_4 = 1$, meaning the displacement from equilibrium in critical damping is

$$\boxed{x(t) = t\exp\left(-\frac{c}{2m}t\right).}$$

Set $m = 1$ kg and $c = 10$ N · s/m and graph $x(t)$ versus t.



Underdamping occurs when $c^2 - 4mk < 0$, that is, when $k = 30$ and $k = 40$. In this case, the solution to equation (2) is

$$r = \frac{-c \pm i\sqrt{4mk - c^2}}{2m}$$

$$r = \left\{\frac{-c - i\sqrt{4mk - c^2}}{2m}, \frac{-c + i\sqrt{4mk - c^2}}{2m}\right\}.$$

Two solutions to the ODE are

$$\exp\left(\frac{-c - i\sqrt{4mk - c^2}}{2m}t\right) \quad \text{and} \quad \exp\left(\frac{-c + i\sqrt{4mk - c^2}}{2m}t\right).$$

By the principle of superposition, then, the general solution to equation (1) is

$$\begin{aligned}
x(t) &= C_5 \exp\left(\frac{-c - i\sqrt{4mk - c^2}}{2m}t\right) + C_6 \exp\left(\frac{-c + i\sqrt{4mk - c^2}}{2m}t\right) \\
&= C_5 e^{-ct/(2m)} \exp\left(-i\frac{\sqrt{4mk - c^2}}{2m}t\right) + C_6 e^{-ct/(2m)} \exp\left(-i\frac{\sqrt{4mk - c^2}}{2m}t\right) \\
&= C_5 e^{-ct/(2m)} \left(\cos\frac{\sqrt{4mk - c^2}}{2m}t - i\sin\frac{\sqrt{4mk - c^2}}{2m}t\right) + C_6 e^{-ct/(2m)} \left(\cos\frac{\sqrt{4mk - c^2}}{2m}t + i\sin\frac{\sqrt{4mk - c^2}}{2m}t\right) \\
&= e^{-ct/(2m)}\left[(C_5 + C_6)\cos\frac{\sqrt{4mk - c^2}}{2m}t + (-iC_5 + iC_6)\sin\frac{\sqrt{4mk - c^2}}{2m}t\right] \\
&= e^{-ct/(2m)}\left(C_7\cos\frac{\sqrt{4mk - c^2}}{2m}t + C_8\sin\frac{\sqrt{4mk - c^2}}{2m}t\right),
\end{aligned}$$

where C_7 and C_8 are arbitrary constants. Differentiate it with respect to t to get the velocity.

$$\begin{aligned}
\frac{dx}{dt} = -\frac{c}{2m}e^{-ct/(2m)} &\left(C_7\cos\frac{\sqrt{4mk - c^2}}{2m}t + C_8\sin\frac{\sqrt{4mk - c^2}}{2m}t\right) \\
&+ e^{-ct/(2m)}\left(-C_7\frac{\sqrt{4mk - c^2}}{2m}\sin\frac{\sqrt{4mk - c^2}}{2m}t + C_8\frac{\sqrt{4mk - c^2}}{2m}\cos\frac{\sqrt{4mk - c^2}}{2m}t\right)
\end{aligned}$$

Apply the initial conditions, $x(0) = 0$ and $x'(0) = 1$, to determine C_7 and C_8.

$$x(0) = C_7 = 0$$

$$\frac{dx}{dt}(0) = -\frac{c}{2m}C_7 + C_8\frac{\sqrt{4mk - c^2}}{2m} = 1$$

Solving this system of equations yields

$$C_7 = 0 \quad \text{and} \quad C_8 = \frac{2m}{\sqrt{4mk - c^2}},$$

meaning the displacement from equilibrium in underdamping is

$$\boxed{x(t) = \frac{2m}{\sqrt{4mk - c^2}}e^{-ct/(2m)}\sin\frac{\sqrt{4mk - c^2}}{2m}t.}$$

Plug in $m = 1$ kg and $c = 10$ N · s/m and graph $x(t)$ versus t for $k = 30$ N/m and $k = 40$ N/m.